LOCKE AND KANT ON MATHEMATICAL KNOWLEDGE

Emily Carson
McGill University, Canada

Both Locke and Kant sought, in different ways, to limit our claims to knowledge in general by comparing it to our knowledge of mathematics. On the one hand, Locke thought it a mistake to think that mathematics alone is capable of demonstrative certainty. He therefore tried to isolate what it is about mathematics that makes it thus capable, in the hope of showing that other areas of inquiry — morality, for example — admit of the same degree of certainty. Kant, on the other hand, attributed much of the metaphysical excess of philosophy to the attempt by metaphysicians to imitate the method of mathematicians. He therefore sought to limit that excess by examining the mathematical method, like Locke, in order to isolate what is special about mathematics that accounts for its certainty.

This paper represents a small part of a larger project relating Kant's views on mathematics to the emergence of the Critical philosophy. Kant's recognition of the role of intuition in mathematics had important implications for his theoretical philosophy. What that role is, and what those implications are, however, are controversial questions. I want to approach these questions via Kant's insistence, even in his pre-Critical work, on a sharp distinction between the method appropriate to mathematics and that appropriate to metaphysics. He often asserted that the application of the mathematical method to metaphysics resulted in flights of dogmatic fancies. An obvious question that arises is *why* the mathematical method issues in genuine knowledge in the one case, but only results in dogmatic fancies in the other. I argue that a philosophically adequate answer to the question raised by Kant's pre-Critical account of the mathematical method requires the Critical doctrine of pure intuition. Understanding the development of Kant's views on mathematics in this way sheds light, I think, on the role of intuition in mathematics for Kant by revealing what questions it is supposed to answer.[1]

E. Carson and R. Huber (eds.), Intuition and the Axiomatic Method, 3–19.

In this paper, I want to compare Kant's pre-Critical account of mathematics with Locke's in order to provide further indirect support for my reading of the role of intuition in the Critical account of mathematics. I argue that Locke offers an account of mathematical knowledge very similar to that offered by Kant in the Prize Essay. In particular, both emphasize what I call the *ideality* of mathematical knowledge in order to explain its peculiar certainty. But there is a tension between this ideality, on the one hand, and what Locke calls the *reality* of such knowledge, on the other. This tension is only resolved by Kant's doctrine of pure intuition. The comparison to Locke is thus intended to bring out the shared concern for the peculiar features of mathematics — that it gives us certain, real, and instructive knowledge — as well as the difficulty of explaining these features. This in turn brings out the importance of Kant's doctrine of pure intuition for such an explanation.

The paper falls into three parts. I will argue, first, that Locke's account was an inadequate representation of the method of attaining mathematical certainty. Secondly, I will present Kant's early account of the mathematical method in the Prize Essay of 1763, in which he presents a somewhat improved version of the Lockean account, but fails to give an adequate philosophical foundation for the resulting view. Finally, I want to show that Kant's notion of pure intuition was the key to the development of an adequate account of the mathematical method and that this, in turn, makes clear why this method was of no use outside of mathematics. I hope to show that Kant's introduction of the notion of construction in pure intuition was thus not simply an application of the independently-developed Critical apparatus, but rather that it is the natural result of philosophical reflection on a shared conception of the mathematical method.

1. Locke on mathematical knowledge[2]

For Locke, there are only two kinds of propositions which we can know with perfect certainty.[3] First there are trifling propositions which have mere verbal certainty and are therefore not instructive: for example, a purely identical proposition like 'A spirit is a spirit', Locke says, does not advance our knowledge. The other propositions which we can know with perfect certainty are those "which affirm something of another, which is a necessary consequence of its precise complex idea, but not contained in it". Locke's stock examples of these propositions are geometrical. For example, since the relation of the outward angle of a triangle to either of the opposite internal angles is no part of the complex idea of triangle, the proposition that the external angle of all triangles is bigger than either of the opposite internal angles is "a real truth, and conveys with it instructive *real knowledge*". Locke contrasts this kind of certain knowledge with our knowledge of substances: since the only access we have to ideas of substances is through our senses, "we cannot make any universal *certain* propositions concerning them". General propositions about substances, insofar as they are certain, are merely trifling; insofar as they are instructive, they are uncertain [4.8.9]. This gives us some idea of why instructive propositions about substances — the subject matter of natural philosophy — cannot be known with certainty. But the question we are concerned with here is why we can have instructive certain knowledge in other areas of inquiry.

Knowledge, for Locke, is the 'perception of the connexion and agreement' of our ideas, where ideas are whatever is the object of our understanding when we think [4.1.1]. The ideas of which we may have certain knowledge, according to Locke, are those whose agreement or disagreement with each other may be intuitively perceived. These ideas also admit of demonstrative certainty. Demonstration, for Locke, is necessary where we cannot compare two such ideas immediately. In that case, the mind must discover the agreement or disagreement of these ideas by means of intermediate ideas. So a demonstration is really a chain of intuitive perceptions of agreement and disagreement among ideas [4.8.3]. The question we have to concern ourselves with now is what is the special feature of such ideas that allows their agreement or disagreement to be intuitively perceived. This requires a brief reminder of Locke's account of ideas.

Our ideas are either simple or complex. Simple ideas, the materials of all our knowledge, are produced in the mind by means of the operation of objects on our senses. Although the mind is "wholly passive" [2.30.3] with respect to simple ideas, once the understanding has stored them, it has the power to "repeat, compare and unite them" to "an almost infinite variety", and thereby "can make at Pleasure new complex Ideas" [2.1.2]. So complex ideas are combinations of simple ideas put together and united under one general name.

Among complex ideas, Locke distinguishes ideas of substance, which are ideas of something "self-subsisting", representing "distinct particular things", from ideas of modes, which are of "dependencies *on*, or affections *of* substance". So ideas of substance are, roughly, ideas of natural kinds of material things, like the idea of lead or of animal. Ideas of modes are best explained in terms of their origin. Whereas ideas of substance, because they purport to refer to self-subsisting things, are combinations of simple ideas which we notice by experience and observation "go constantly together", ideas of modes are *voluntary* combinations of ideas. Locke emphasises the free activity of the mind, its power to repeat and join its own ideas "as it pleases" and "without the help of any extrinsical Object, or any foreign suggestion" [2.13.1]. The simple ideas of which ideas of modes are composed are not to be thought of as characteristic marks of any real beings with "steady existence", but as "scattered and independent Ideas put together by the Mind". Unlike ideas of substances, they have their origin and existence "more in the Thoughts of Men, than in the reality of things". To form such ideas "it sufficed that the Mind put the parts of them together and that they were consistent in the Understanding, without considering whether they had any real Being". For example, although the idea of man has no more connection in nature with the idea of killing than does the idea of sheep, we combine the ideas of man and of killing, and make it into a species of action, signified by the word ‘murder’.

It is this difference, according to Locke, which underlies the difference between demonstrative sciences like mathematics, and natural philosophy. Ideas of modes, the subject matter of mathematics, are just combinations of simple ideas which the mind puts together arbitrarily, of its own choice, without reference to any “real existence”, and subject only to the condition that the simple ideas be “consistent in the understanding”. Because they have their existence “in the thoughts of men” rather than “in the reality of things”, we can have perfect knowledge of them. On the other hand, ideas of substance, the subject matter of natural philosophy, purport to refer to things as they really exist, and to represent that constitution on which all their properties depend. Thus we can never be sure that we have captured all the various qualities belonging to the thing.

Locke formulates the difference in terms of his conception of essence. Recall that for Locke, real essence is “the real internal, but generally in substances, unknown constitution of things, whereon their discoverable qualities depend” [3.3.15]. Nominal essence is not the real constitution of things, but rather “the *artificial* constitution of genus and species”; it is the “workmanship of the understanding” in ranking things into sorts. In the case of substances, the real essence is different from the nominal essence. The real essence is the unknown, perhaps corpuscular, constitution of a substance, while the nominal essence is a combination of perceptible properties of the substance. In the case of modes, however, the real essence and the nominal essence are the same. Both are the ‘workmanship of the mind’, ‘creatures of the understanding’. To illustrate the difference, Locke compares the idea of triangle with that of gold:

> . . . a Figure including a Space between three Lines, is the real, as well as nominal *Essence* of a Triangle; it being not only the abstract *Idea* to which the general Name is annexed, but the very *Essentia*, or Being, of the thing itself, that Foundation from which all its Properties flow, and to which they are all inseparably annexed.

In the case of gold, however, the real essence is

> . . . the real Constitution of its insensible Parts, on which depend all those Properties of Colour, Weight, Fusibility, Fixedness, etc. which are to be found in it. Which Constitution we know not; and so having no particular Idea of, have no Name that is the Sign of it. But yet it is its Colour, Weight, Fusibility, and Fixedness, etc. which makes it to be *Gold*, or gives it a right to that Name, which is therefore its nominal Essence [3.3.18].

This, finally, is the key difference between ideas of modes and of substances which explains why we have certain knowledge of the one, but not of the other. Because ideas of modes are combinations of ideas which the mind puts together arbitrarily without reference to any real existence outside it, the real essence *just is* the nominal essence. Because we know the nominal essence (we create it), ideas of modes have knowable *real* essences. Ideas of substances do not.

It is upon this ground, Locke says, that he is "bold to think"

> ...that *Morality is capable of Demonstration*, as well as Mathematicks: Since the precise real Essence of the Things moral Words stand for, may be perfectly known; and so the Congruity, or Incongruity of the Things themselves, be certainly discovered, in which consists perfect Knowledge [3.11.16]

In summary, then, it seems that the relevant difference between ideas of substance and ideas of modes is that ideas of modes are in some sense purely *ideal*. Mathematics is, Locke says, "only of our own *Ideas*" [4.4.6]. Discourses about morality are "about *Ideas* in the mind ... having no external Beings for *Archetypes* which they ... must correspond with" [3.11.17]. Two things follow from the ideality of ideas of modes. First, because they do not refer to anything outside the mind, "they have no other reality but what they have in the minds of men", they "have the perfection that the mind intended them to have". For example, the idea of a figure with three sides meeting at three angles is, Locke says, a complete idea, requiring nothing else to make it perfect; it contains "all that is, or can be essential to it, or necessary to complete it, wherever or however it exists". So complete knowledge of the idea amounts to complete knowledge of the object of the idea. Secondly, Locke seems to think that it follows from the fact that these ideas are 'the Workmanship of the Understanding' that they are transparent to us: thus, we can have complete knowledge of ideas of modes, and thus of the objects of those ideas. Reformulated in terms of essences, the two important consequences of Locke's thesis of the ideality of ideas of modes are (1) that the real essence is the same as the nominal essence, and (2) that the real essence is (therefore) knowable.

This immediately gives rise to a question regarding what Locke calls the 'reality' of such ideas and of our knowledge of them. The problem is that if knowledge consists only in the perception of the agreement or disagreement of our own ideas, then it seems that regardless of "how things are", the reasoning of a wise person will be just as certain as the most extravagant fancies of an "enthusiast". Ideas which are mere chimera may be spoken of consistently and coherently, and thus "such Castles in the Air, will be as strong Holds of Truth, as the Demonstrations of *Euclid*" [4.4.1]. But what value or use is there in such knowledge of our own imaginations? For what "gives value to our Reasonings, and preference to one Man's Knowledge over another's, [is] that it is of Things as they really are, and not of Dreams or Fancies" [4.4.1].

Although Locke raises this question with regard to knowledge in general, it seems particularly pressing with respect to his account of our ideas of modes. The content of ideas of substances is easily explained by their reference to "extrinsical objects". But if ideas of modes really are just voluntary collections of ideas, put together without any consideration as to whether they have 'real being', then how can they provide us with real knowledge? What makes our reasoning about squares and circles count as knowledge where our reasoning about mere chimera like harpies and centaurs fails to count as knowledge? How do we distinguish mathematical theories from mere fairy tales about castles in the air? More to the point, how do we distinguish our knowledge of triangles from our knowledge of two-sided rectilinear figures?

The problem is that, at least in the case of geometry, we don't want to say that *any* combination of simple ideas results in a real idea of a mode. But how does Locke rule out, say, ideas of two-sided rectilinear figures? He says that ideas of modes must be "consistent in the understanding", but what does this mean? Is it mere logical consistency? We find the answer to this question in Locke's account of the various modes of space, where he explains how our ideas of geometrical figures are generated by the activity of the mind "repeating its own Ideas and joining them as it pleases". He describes the power of the mind to join lines of whatever length to other lines of different lengths and at different angles until it encloses a space, and thereby multiply figures both in their size and shape *in infinitum*. These products of the mind are the subject matter of geometry. What Locke describes here as the generation of complex geometrical ideas is the construction of figures against a spatial background. So it turns out that geometrical ideas have spatial properties built into them. It's not that the idea of the two-sided rectilinear figure is *logically* inconsistent: it is rather that those simple ideas cannot be combined *in this way*, in space.

This reading fits with Locke's example of a geometrical demonstration: since the mind cannot compare the sizes of three angles of a triangle and two right angles immediately, it finds some other angles to which the three angles of the triangle are equal, and then determines that those are equal to two right ones, thereby coming to know the equality of the three angles to two right ones. Ideas here then are to be taken as quasi-sensible images.

The *problem* with this reading is that it *doesn't* fit with Locke's two claims based on the ideality of modes, the two claims that I suggested are essential to his account of why these ideas admit of complete certainty. This problem comes out most clearly when we consider Locke's claim that mathematics is not the only science capable of demonstrative certainty. In particular, he thinks that our ideas of a supreme being and of ourselves are clear enough to provide "such Foundations of our Duty and Rules of Action as might place *Morality amongst the Sciences capable of Demonstration*" [4.3.18]. He proceeds to back up this claim by arguing that the proposition that where there is no property, there is no injustice "is a Proposition as certain as any Demonstration in *Euclid*". The idea of property, he says, is a right to anything; the idea to which the name 'justice' is attached is the invasion or violation of that right. With these names attached to these ideas, the proposition can be known to be true as certainly as any proposition in mathematics. Similarly, if the idea of government is the establishment of society upon certain rules which demand conformity, and the idea of absolute liberty is for any one to do whatever he pleases, then we know with demonstrative certainty the truth of the proposition that no government allows absolute liberty.

These examples lead one naturally to question the strength of the analogy with geometrical demonstrations. What is striking about geometrical examples, and what struck Locke, is how they provide us with certain, real and *instructive* knowledge. The same cannot be said of Locke's examples of demonstrated ethical truths. As Locke presents them, these so-called demonstrations seem to involve nothing more than analyses of complex concepts into their simple constituents. At least one contemporary critic, Henry Lee, pointed out that these demonstrations seem to result only in trifling or vacuous propositions of no use to us.[4] On the other hand, they do seem to fit better with Locke's account of the ideality of modes: these ethical ideas really are arbitrary combinations of simple ideas. They *do* seem amenable to certain knowledge in virtue of their transparency: we put them together out of simple ideas, and we can therefore break them down to those simple ideas and recombine them. But the resulting propositions are not instructive in the way that geometrical propositions are.

As we've seen, geometrical ideas are *not* arbitrary combinations in the same sense. There are some combinations of simple ideas such that if the mind tries to combine them, it will fail. There are external non-logical constraints on the combination of the simple ideas of space: that combination is therefore governed by rules in a way in which the combination of simple ethical ideas appears not to be. The problem with this is that, as I argued above, Locke's claim that we have certain knowledge of these ideas rests on their ideality: on the fact that they are only ideas in our mind, referring to nothing outside our minds, that we know their real essences. This is supposed to give us privileged epistemic access to them. But the ideas of geometry are not ideal in the relevant sense. The fact that there are external 'extra-mental' constraints, that space, in effect, acts as a background theory, shows that this is not the case. In short, Locke's account of the *content* of mathematics conflicts with his account of the *certainty* of mathematics.

Locke does acknowledge *a* disanalogy between geometrical demonstration and ethical demonstration in that there is a special role for figures in geometry. But he takes this to be a merely heuristic role: figures are "helps to the memory" in retaining the many ideas involved in any given demonstration; because the diagrams are copies of the ideas, they have a greater correspondence with the ideas than do words or sounds. What he fails to see is that the role of the diagram reflects an *essential* difference between these kinds of ideas. On the contrary, he thinks that this disadvantage to ethical ideas can easily be overcome by means of definitions: we simply set down the collection of simple ideas which every term shall stand for and then use the term steadily and constantly for that precise collection. Then presumably, to perceive that two ideas agree involves something like running through the list of simple ideas contained in each. It is hard to see how this results in anything more than what Locke calls trifling knowledge, particularly since we put those ideas there in the first place. The difference in the kinds of demonstration appropriate to ethics and to geometry could not be made clearer.

In summary, then, Locke's account of modes is supposed to capture what is different between demonstrative sciences like mathematics and ethics on the one hand, and natural philosophy on the other. Because mathematics and ethics are only of our own ideas, we have certain demonstrative knowledge of them. I have tried to suggest (i) that insofar as geometry is a body of demonstratively certain instructive truths, as Locke describes it, its objects are not arbitrary creations of the mind, and (ii) insofar as the objects of ethics are arbitrary creations of the mind, it does not admit of demonstrative certainty of instructive truths in the way that geometry does. The problem here lies with Locke's failure to account for the essential role of spatial constructions in geometrical demonstration. More important than the failure of Locke's analogy between mathematics and ethics, however, is that his account of mathematical knowledge is radically incomplete. The claim is supposed to be that we have privileged knowledge of the properties of geometrical figures because they are only in our minds, we know the real essences. The fact that there are external 'extra-mental' constraints — that space, in effect, acts as a background theory — shows that this cannot be the case. In failing to integrate these constraints on the generation of geometrical ideas and our knowledge of these constraints into his account of demonstrative certainty, Locke has failed to explain how we come to have demonstrative certainty even in mathematics.

2. Kant on the method of mathematics

In the Prize Essay of 1763, Kant takes up the question of whether metaphysics is capable of the same degree of certainty as mathematics. Like Locke, he examines the method of mathematics to determine whether it can be applied in areas other than mathematics. Unlike Locke, however, his conclusion is negative, at least with respect to metaphysics. Indeed, Kant thinks that nothing has been more damaging to philosophy than the imitation of the mathematical method.

The primary difference in method that Kant considers in the Prize Essay concerns the role of definition. This difference is summarised in the following passage:

> In mathematics I begin with the definition of my object, for example, of a triangle or a circle, or whatever. In metaphysics I may never begin with a definition. Far from being the first thing I know about the object, the definition is nearly always the last thing I come to know. In mathematics, namely, I have no concept of my object at all until it is furnished by the definition. In metaphysics I have a concept which is already given to me although it is a confused one. My task is to search for the distinct, complete and determinate concept.[5]

This is made possible by the fact that "mathematics arrives at all its definitions synthetically, whereas philosophy arrives at its definitions analytically" (2:276). A synthetic definition, according to Kant, is arrived at by "the arbitrary combination of concepts". The concept thus defined is not given prior to the definition, but rather "comes into existence" as a result of the definition. For example,

> [w]hatever the concept of cone may ordinarily signify, in mathematics the concept is the product of the arbitrary representation of a right-angled triangle which is rotated on one of its sides [2:276].

To take another example, the concept of a square is the result of the arbitrary combination of the concepts *four-sided, equilateral*, and *rectangle*.[6] This is not the result of an analysis of some concept given in another way — it is not, for example, abstracted from our experience of squares in nature; the concept is, as Kant says, first given by the definition itself.

In philosophy, on the other hand, the concepts are always *given* in some way, but "confusedly or in an insufficiently determinate fashion". The task of the philosopher is then to discover by means of analysis the characteristic marks in the confused concept in order to arrive at a complete and determinate concept, that is, a definition. Thus Kant says for example, "everyone has the concept of time". This idea that everyone has must be examined in all kinds of relations, and once the characteristic marks have been made distinct, and then combined together, the resulting concept has to be compared with the concept of time which was originally given in order to determine whether or not it has captured the original idea. If by contrast we tried to arrive at a definition of time synthetically, by arbitrarily combining concepts, it would have been a "happy coincidence" if the resulting concept had been exactly the same as the idea of time which is given to us [2:277]. So in mathematics, we produce concepts by means of synthetic definitions. In philosophy, we analyse *given* concepts in order to arrive at analytic definitions.

In order to appreciate the importance of this methodological distinction, we have to consider Kant's theory of definition in a bit more detail. He elaborates on this in his lectures on logic from the early 1770's. A definition is essentially a distinct and complete concept of a thing.[7] A concept is *distinct* insofar as one is *conscious* of the marks contained in the concept [24:120]. A concept is *complete* when the marks are sufficient to cognise, first, the difference of the *definitum* from all other things, and secondly, the identity of it with other things.

Kant claims that the synthetic definitions of mathematics are definitions in this sense, and in fact, that mathematical concepts are the only ones that admit of definition. First of all, he says, all fabricated concepts are "produced simultaneously with their distinctness": I am conscious of each of the marks included in the concept because I put them there in defining the concept, and "one can most easily be conscious of that which one has oneself invented" [24:153]. Similarly, the definition is complete because the mathematician

> . . . thinks everything that suffices to distinguish the thing from all others, for [it] is not a thing outside him, which he has cognised in part according to certain determinations, but rather a thing in his pure reason, which he thinks of arbitrarily and in conformity with which he attaches certain determinations, whereby

> he intends that the thing should be capable of being differentiated from all other things [24:125].

In other words, if the thing defined is first *given* by the definition, then the definition is of course complete.

This is in sharp contrast to empirical concepts which, Kant says, are capable only of description, not of definition. Since in that case the concept is given, in order to make it distinct I must "enumerate all the marks that I *think* in connection with the expression of the *definitum*". But one can never know that the marks that one has enumerated at any point are "sufficient to distinguish the thing from all remaining things" [24:124]. The most we can hope for is comparative completeness, "when the marks of a thing suffice to distinguish it from everything that we have cognised in experience until now".

Since philosophical concepts, like empirical ones, are also given, "the philosopher cannot so easily be certain that he has touched on all the marks that belong to a thing, and that he has insight into these completely perfectly"; consequently, many marks "may still belong to the thing of which he knows nothing" [24:153]. This suggests that philosophical concepts, like empirical ones, do not in the end admit of definition either. At best, any purported definition will be uncertain.

It seems, then, that Kant's account of the certainty of mathematical knowledge shares the essential features of Locke's account. What is special about mathematical concepts is that they are given by synthetic definitions — by the arbitrary combination of concepts. Because I defined the concept, I am conscious of each of the marks included in it; because the thing defined is not a thing outside me, but is first given by the definition, then all the marks which I include in the definition of the thing are all the marks that *belong* to the thing. In other words, to explain the certainty of mathematics, Kant, like Locke, appeals to what I called earlier the 'ideality' of mathematical concepts: because we make the concepts of mathematics, we have perfect insight into them.

Just as Locke expresses this in terms of his doctrine of real and nominal essences — claiming that since in the case of ideas of modes, the nominal essence is also the real essence, it follows that we can have knowledge of the real essences — Kant, in the lectures on logic, distinguishes between real and nominal *definitions*. A definition is *nominal* when its marks are "adequate to the whole concept *that we think* with the expression of the *definitum*"; a *real* definition is one "whose marks constitute the whole possible concept of the thing" [24:132]. Alternatively, nominal definitions "contain everything that is equal to the whole concept *that we make for ourselves of the thing*", whereas real definitions "contain everything that belongs to the thing in itself". In particular, Kant says that all definitions of arbitrary concepts that are made, as opposed to given, are real definitions

> ...because it lies solely with me to make up the concept and to establish it as it pleases me, and the whole concept thus has no other reality than merely what my fabrication wants; consequently I can always put all the parts that I name

> into a thing, and these must then constitute the complete, possible concept of the thing, for the whole thing is actual only by means of my will [24:268].

Empirical concepts, on the other hand, would be capable of at best nominal definition since "I do not define the object but instead only the concept that one thinks in the case of the thing" [24:271]. The difference is that in the case of arbitrary concepts, the marks of the "whole possible concept of the thing" just are the marks of the concept that we *think* in the case of the thing: in defining the concept that one thinks, one at the same defines the object. Because mathematical definitions are of arbitrary concepts, they are also, by Kant's lights, *real* definitions.

Kant attributes much mistaken philosophy to the failure to recognise this fundamental methodological difference between philosophy and mathematics: that the one arrives at its definitions by analysis, the other by synthesis. Indeed, it underlies his diagnosis in the Prize Essay of the main problem of philosophy: "nothing has been more damaging to philosophy", he says, than the imitation of the method of mathematics "in contexts where it cannot possibly be employed" [2:283]. For example, a philosopher could offer a synthetic definition by "arbitrarily thinking of a substance endowed with a faculty of reason and calling it a spirit". However, this would not be a *philosophical* definition, but a "grammatical" one, a mere linguistic stipulation, and "no philosophy is needed to say what name is to be attached to an arbitrary concept" [2:277]. Indeed, Kant accuses Leibniz of having made this mistake in imagining "a simple substance which had nothing but obscure representations" and calling it a 'slumbering monad'. He did not thereby *define* the monad, "he merely invented it, for the concept of a monad was not given to him but created by him".

This charge immediately gives rise to the question of what licenses this way of drawing the distinction between mathematics and philosophy. Kant claims that his treatise contains nothing but "empirical propositions", a neutral description of the different methods appropriate to mathematics and to philosophy. But for his *pre*scription against the application of the mathematical method in philosophy to carry any weight, he owes us an account of *why* the mathematical method is appropriate in the one and not the other. What is the relevant difference between mathematical concepts and metaphysical ones, a difference which accounts for the admissibility of arbitrary concepts in the one but not in the other: why is invention permissible, even required, in mathematics, but not in philosophy? Why is the synthetic definition of a trapezium legitimate, and Leibniz's invention of the slumbering monad not? After all, both seem to involve the formation of complex concepts from given primitive ones. More to the point, the question arises for Kant just as it did for Locke: why does the synthetic definition of a trapezium issue in a legitimate mathematical concept, while the definition of a figure enclosed by two straight lines does not?

In short, Kant is subject to Locke's worry that reasoning about 'Castles in the Air' will be 'as strong Holds of Truth as the Demonstrations of *Euclid*', but Kant has somewhat more to say on this matter than Locke does. The comparison of the role of definitions in metaphysics and mathematics is only part of the general comparison of their respective methods as presented in the Prize Essay. For one thing, unlike Locke, Kant recognises a role for 'indemonstrable propositions' in mathematics. Even if they admit of proof elsewhere, he says, "they are nonetheless regarded as immediately certain in this science" [2:281]. These propositions are set up at the beginning of mathematical inquiry "so that it is clear that these are the only obvious propositions which are immediately presupposed as true".

So the following picture emerges from the Prize Essay of the method of mathematics. It begins with a few given concepts, which mathematicians cannot and must not define, such as magnitude in general, unity, plurality and space, and a small number of indemonstrable, immediately certain propositions, such as the propositions that the whole is equal to all its parts taken together, and that there can only be one straight line between two points. Further concepts are built up out of the given ones by arbitrary combination — by synthesis — in accordance with the fundamental propositions. The mathematician then derives further propositions from these complex concepts and the fundamental propositions. Kant has made some progress over Locke because he explicitly distinguishes mathematical demonstration from conceptual analysis; he says little, however, about how the theorems *are* derived from the complex concepts and fundamental propositions.

Nonetheless, we have here the beginnings of an answer to the question about how to rule out the figure enclosed by two straight lines: the figure cannot be defined in accordance with the indemonstrable propositions, for it contradicts the proposition that between two points only one straight line may be drawn. The indemonstrable propositions therefore place constraints on the arbitrary combination of concepts. But this then simply pushes the question onto the indemonstrable propositions. What is *their* status? It is not enough to *presuppose* them as true; they must in fact be true, and be known to be true if we are to distinguish the demonstrations of Euclid from mere castles in the air. Kant says that they are immediately certain, but what makes them so?

The problem here is that Kant's description of the mathematical method seems to correspond roughly (the technique of derivation aside) to that appropriate to a formal axiomatic system. But unless some explanation is given of the content of those primitive concepts and propositions and the ground of their certainty, this account collapses into a kind of formalism. Regardless of Kant's views about formalism as a philosophy of mathematics (and it's clear that he opposes it), the threat of formalism undermines his attempt to distinguish the methods appropriate to mathematics and metaphysics. If the geometer is simply deducing properties and relations of imaginary or *ideal* objects given by arbitrary definitions, what is to stop the metaphysician from developing an axiomatic system for slumbering monads in a similar way? In what sense can we say that mathematics is a body of *truths*, and the theory of slumbering monads is not? More importantly, given Kant's concern with the relative certainty of mathematics and metaphysics, how can we say that we know these truths with certainty?

To sum up then, Kant's account of mathematics in the Prize Essay seems to leave open the question of the relation between the method of mathematics, its content, and its certainty. First of all, it's not clear how mathematical concepts are anything but arbitrary inventions with no objective content; secondly, mathematical propositions then seem to lose their claims to truth as opposed to mere deducibility from axioms and definitions; and thirdly, it's therefore not clear that Kant is entitled to the sharp distinction he wants to draw between the certainty of mathematics and that of metaphysics. The key to all of these questions with respect to geometry is the relationship between geometry and space. Like Locke, Kant does recognise a role for symbols in mathematical proofs — drawn figures, in the case of geometry — as "an important device which facilitates thought". The examples of geometrical proofs in the Prize Essay are clearly diagrammatic proofs. But, again like Locke, he fails to integrate this feature into the general account of the method of mathematics: he tells us that "figures and visible signs" play a role in mathematical proofs, but he fails to explain what that role is and how they fulfill it. Considering that his goal is to contrast the nature of mathematical and philosophical certainty, it would seem that he owes us an account of why the distinguishing features of mathematics are guarantees of the *certainty* of mathematics. It seems then that the important task is to provide an epistemological grounding for the mathematical method. He has to show that the mathematical method of attaining certainty is in fact a method of attaining certainty.

3. Construction in pure intuition

Kant undertakes this task in the *Critique of Pure Reason*. In 'The discipline of pure reason in its dogmatic employment' near the end of the *Critique*, he once again takes up the question of whether the mathematical method of attaining certainty is identical with the method of attaining certainty in philosophy. The answer, again, is negative, but the reasons appear different. The essential difference between these two kinds of knowledge is again a formal difference, that "philosophical knowledge is the knowledge gained by reason from concepts" whereas "mathematical knowledge is the knowledge gained by reason from the *construction* of concepts".[8] To construct a concept is, for Kant, to "exhibit apriori the intuition which corresponds to the concept". To take one of his examples, the geometer constructs a triangle "by representing the object which corresponds to this concept either by imagination alone, in pure intuition, or in accordance therewith also on paper, in empirical intuition — in both cases completely apriori, without having borrowed the pattern from any experience". Although the intuition is a single object, "it expresses universal validity for all possible intuitions which fall under the same concept" because it is "determined by certain universal conditions of construction" [A714/B742]. Contrast with this a philosophical concept, like that of cause or reality. "No one can obtain an intuition corresponding to the concept of reality otherwise than from experience; we can never come into possession of it apriori out of our own resources, and prior to the empirical consciousness of reality" [A714/B742].

So far, this is again just a description of the difference: objects corresponding to mathematical concepts can be provided a priori, but this is not the case in metaphysics. Recognising the need for an *explanation*, though, Kant asks "what can be the reason of this radical difference in the fortunes of the philosopher and the mathematician, both of whom practise the art of reason, the one making his way by means of concepts, the other by means of intuitions which he exhibits apriori in accordance with concepts?" Why is it possible for mathematicians to obtain a priori intuitions corresponding to their concepts, but not for philosophers?

The answer, according to Kant, is given by the "fundamental transcendental doctrines" which he has just elaborated. According to these doctrines, there are two elements in the field of appearance: the form of intuition (space and time), and the matter (the physical element). Whereas the material element cannot be given in determinate fashion other than empirically, the formal element, Kant says, "can be known and determined completely apriori". What does this mean? Objects are given only through intuition. The only intuition given a priori, as Kant argued in the Transcendental Aesthetic, is that of the form of appearances. Because space and time, as the form of appearances, are given a priori, "a concept of space and time as quanta, can be exhibited apriori in intuition, that is, constructed" [A720/B748]. Objects for philosophical concepts, such as that of reality or substance, however, can only be given in empirical intuition, aposteriori. So, Kant concludes, corresponding to two elements in the field of appearance, there is a twofold employment of reason: the mathematical employment of reason through the construction of concepts, and the philosophical employment of reason in accordance with concepts.

So it's clear that the doctrine of pure intuition accounts for the difference between these two methods. But what does this tell us about the *role* of intuition in the method of mathematics? To answer this question, I want to consider how the account of mathematical method in the *Critique of Pure Reason* relates to the earlier account given in the Prize Essay. In the Prize Essay, the difference between mathematics and philosophy, between the synthetic and the analytic method, rested largely on the different roles of definitions in each. Similarly in the *Critique*, Kant attempts to show once and for all that mathematics and philosophy are so different that "the procedure of the one can never be imitated by the other". He does this by once again considering the means of achieving certainty in mathematics — that is, "definitions, axioms, and demonstrations" — and showing that "none of these, in the sense in which they are understood by the mathematician, can be achieved or imitated by the philosopher" [A726/B754]. I'll begin with the account of definition in the *Critique*.

Again, Kant says that to define means to present the complete and distinct concept of a thing. An empirical concept cannot be defined because the limits of the concept are never assured: for example, new observations remove some properties and add others. Concepts given a priori (such as substance or cause) cannot be defined because the completeness of the analysis will always be only probably, never apodeictally, certain. The only concepts which remain are "arbitrarily invented concepts". With regard to these, Kant says,

> . . . a concept which I have invented I can always define; for since it is not given to me either by the nature of understanding or by experience, but is such as I have myself deliberately made it to be, I must know what I have intended to think in using it [A729/B757].

However, he goes on, "I have [not] thereby defined a true object". If the concept depends on any empirical conditions, "this arbitrary concept of mine does not assure me of the existence or of the possibility of its object". To borrow Kant's phrase from the Prize Essay, it would just be a "happy accident" if there were an object corresponding to my invented concept. Mathematical concepts, on the other hand, "contain an arbitrary synthesis that admits of apriori construction". The constructibility of such concepts in pure intuition assures us of the possibility of the corresponding object. Consequently, only mathematics has definitions proper "for the object which it thinks, it exhibits apriori in intuition, and this object certainly cannot contain either more or less than the concept, since it is through the definition that the concept of the object is given" [A729/B757].

Herein lies the reason why synthetic definitions are admissible in mathematics and not in philosophy: the arbitrary combination of concepts in mathematics admits of a priori construction, which assures us of the existence, or better, the possibility of the objects. It is in this sense, then, that mathematical definitions are also real definitions: a real definition, Kant says, "does not merely substitute for the name of a thing other more intelligible words, but contains a clear property by which the defined *object* can always be known with certainty" [A242n]. Thus, Kant says earlier in the *Critique*, a real definition "makes clear not only the concept but also its *objective reality*". Because mathematical definitions present the object in intuition, in conformity with the concept originally framed by the mind, they are real definitions. Mathematical definitions *are*, Kant says, constructions of concepts [A730/B758].

What this amounts to, I think, is a rejection of what I have called the ideality thesis. As a result, Kant is able to distinguish explicitly the two models of demonstration that I suggested Locke conflates in his attempt to uphold the ideality thesis. Kant explains that mathematics is not concerned with analytic propositions because as a mathematician, "I must not restrict my attention to what I am actually thinking in my concept of a triangle"; "this" he says, "is nothing more than the *mere definition*". Similarly, he goes on to say that it would be futile for the mathematician to philosophise upon the triangle, to think about it discursively, for "I should not be able to advance a single step beyond the mere definition, which is what I had to begin with".

In order to gain mathematical knowledge, instead "I must pass beyond this definition to properties which are not contained in this concept *but yet belong to it*". The only way to do this is to "determine my object in accordance with the conditions ... of pure intuition" [A718/B746]: to construct it. So whereas Locke gave no explanation of how properties not contained in the mathematical concept nonetheless belonged to it, Kant claims that they do so in virtue of the concept's relation to the conditions of pure intuition: "we can determine our concepts in apriori intuition, inasmuch as we create for ourselves, *in space and time* ... the objects themselves". The doctrine of the form of intuition thus enables Kant to integrate the appeal to spatial construction into his account of the mathematical method.

Let me try to make clear now how the doctrine of pure intuition resolves the problems raised above regarding Kant's early account of mathematical method. That Kant was worried about the formalist possibility left open in the Prize Essay, and that he saw the doctrine of pure intuition as eliminating the possibility, comes out clearly in many passages in the *Critique*. He says, for example, that geometrical knowledge would be nothing but playing with mere chimeras "were it not that space has to be regarded as a condition of the appearances which constitute the material for outer experience" [A157/B196]. The key idea here is that the content of the arbitrary concepts of mathematics is given a priori, by construction in pure intuition, whereas there is nothing given a priori which could correspond to the concept of a slumbering monad. Pure intuition thus constrains the arbitrariness of the definitions and gives content to the axioms and primitive concepts. The fundamental propositions of geometry assert the "universal conditions of construction" of figures, that is, the conditions imposed by the form of intuition. Construction in pure intuition is then simply construction according to the (Euclidean) postulates. The concept of a figure enclosed by two straight lines is not in accordance with the fundamental propositions (i.e., between any two points only one straight line can be drawn), and thus is *not* constructible in pure intuition. No similar constraints can be prescribed in advance regarding the existence of the objects corresponding to philosophical concepts.

Note, though, that this account requires that those constraints be prescribed in advance. What we've got so far is a story about how the fundamental propositions and arbitrary concepts of mathematics have objective content. The mathematician does not simply spin out consequences of arbitrary theories, but rather spins out consequences of *true* theories. But it's not enough that these theories simply be presupposed as true; they must be *known* to be true. Otherwise the door is left open for the axiomatic theory of slumbering monads. For Locke and Kant's earlier self, the ideality thesis was supposed to explain how we do have certain knowledge of mathematics. Because Kant has rejected the ideality thesis, he now has to provide another explanation of the certainty of mathematics. It is for this reason, I want to suggest, that Kant must see a role for his doctrine of intuition in explaining not only the content, but our knowledge, of the indemonstrable propositions of mathematics. Only in this way can he achieve what Locke and Kant's earlier self couldn't: a coherent account both of the content of mathematics and of the certainty of our knowledge of mathematics.

Notes

1. I argue for this in detail in Carson (1999).
2. This section draws on my paper Carson (2002).
3. Locke, *An Essay Concerning Human Understanding*, 4.8.8.
4. Cited in Schneewind, J., "Locke's Moral Philosophy" in Chappell, Vere (ed.), *The Cambridge Companion to Locke*. Cambridge University Press, 1994; p. 223.
5. *Untersuchung über die Deutlichkeit der Grundsätze der natürlichen Theologie und der Moral*, 2:283.
6. Dohna-Wundlacken logic, 24:757.
7. Blomberg logic, 24:263.
8. *Critique of Pure Reason*, A713/B742.

References

References to the *Critique of Pure Reason* are given by the standard pagination of the A and B editions. References to Kant's other writings are given by volume and page number of the Akademie edition of *Kants gesammelte Schriften*. The English citations from the *Critique of Pure Reason* follow the translation of Paul Guyer and Allen Wood in the *Cambridge Edition of the Works of Immanuel Kant*. Citations from the pre-Critical works follow the translation of David Walford in collaboration with Ralf Meerbote in *Theoretical Philosophy 1755–1770*. Citations from Kant's lectures on logic follow J. Michael Young's translation in the *Cambridge Edition* volume.

Carson, E. (1999), "Kant on the Method of Mathematics" in: *Journal of the History of Philosophy* **37** (4), 629–652.

Carson, E. (2002), "Locke's Account of Certain and Demonstrative Knowledge" in: *British Journal for the History of Philosophy* **10** (3), 359–378.

Kant, I. (1992a), *Lectures on Logic*, ed. by J. Michael Young, Cambridge University Press.

Kant, I. (1992b), *Theoretical Philosophy 1755–1770*, translated and edited by D. Walford, Cambridge University Press.

Kant, I. (1998), *Critique of Pure Reason*, translated by P. Guyer and A. Wood, Cambridge University Press.

Locke, J. (1975), *An Essay Concerning Human Understanding*, ed. by P. H. Nidditch, Clarendon, Oxford.

Schneewind, J. (1994), "Locke's Moral Philosophy" in: V. Chappell (ed.), *The Cambridge Companion to Locke*, Cambridge University Press, 199–225.